

On a septuple product identity

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For polynomials f and g in the variable n with integer coefficients let us define

$$\Omega(f) = \sum_{n=-\infty}^{\infty} (-1)^n q^{f(n)}$$

and

$$\Omega(f, g) = \sum_{n=-\infty}^{\infty} (-1)^n q^{f(n)} x^{g(n)}.$$

In [1] Farkas and Kra state and prove the following identity

Theorem 1

$$\begin{aligned} & \prod_{n=1}^{\infty} (1 - q^{2n})^2 (1 - xq^{2n-2}) (1 - x^{-1}q^{2n}) (1 - x^2q^{4n-2}) \\ & \times (1 - x^{-2}q^{4n-2}) (1 - x^2q^{4n-4}) (1 - x^{-2}q^{4n}) \\ = & \Omega(5n^2 + n)(\Omega(5n^2 + 3n, 5n + 3) + \Omega(5n^2 - 3n, 5n)) \\ & - \Omega(5n^2 + 3n)(\Omega(5n^2 + n, 5n + 2) + \Omega(5n^2 - n, 5n + 1)). \end{aligned}$$

□

Their proof exploits identities involving theta functions. Foata and Han [2] gave a more elementary deduction of (1) from Jacobi's triple product formula and Watson's quintuple product formula. Here we give a still more elementary derivation just from Jacobi's triple product formula.

Let P denote the product on the of the identity in Theorem 1. First observe that we can rewrite P as

$$\prod_{n=1}^{\infty} (1 - q^{2n})^2 (1 - xq^{2n-2}) (1 - x^{-1}q^{2n}) (1 - x^2q^{2n-2}) (1 - x^{-2}q^{2n}). \quad (1)$$

We now use Jacobi's triple product formula in the form

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 - xq^{2n-2})(1 - x^{-1}q^{2n}) = \sum_{k=-\infty}^{\infty} (-1)^k x^k q^{k^2-k}. \quad (2)$$

We obtain (2) by replacing q by q^2 and k by $-k$ in formula (1.1) of [2]. Replacing x by x^2 in (2) gives

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 - x^2 q^{2n-2})(1 - x^{-2} q^{2n}) = \sum_{k=-\infty}^{\infty} (-1)^k x^{2k} q^{k^2-k}. \quad (3)$$

Multiplying (2) by (3) and using (1) gives

$$P = \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} x^{k+2l} q^{k^2-k+l^2-l}.$$

Thus

$$P = \sum_{r=-\infty}^{\infty} a_r(q) x^r$$

where

$$a_r(q) = \sum_{k+2l=r} (-1)^{k+l} q^{k^2-k+l^2-l} = \sum_{l=-\infty}^{\infty} (-1)^{r-l} q^{5l^2-(4r-1)l+r^2-r}.$$

We now evaluate $a_r(q)$ by dividing into cases according to the residue of r modulo 5. If $r = 5m$ then

$$5l^2 - (4r - 1)l + r^2 - r = 5(l - 2m)^2 + (l - 2m) + 5m^2 - 3m$$

and so

$$a_{5m}(q) = (-1)^m q^{5m^2-3m} \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+n} = (-1)^m q^{5m^2-3m} \Omega(5n^2 + n).$$

If $r = 5m + 1$ then

$$5l^2 - (4r - 1)l + r^2 - r = 5(2m - l)^2 + 3(2m - l) + 5m^2 - m$$

and so

$$a_{5m+1}(q) = -(-1)^m q^{5m^2-m} \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+3n} = -(-1)^m q^{5m^2-m} \Omega(5n^2 + 3n).$$

If $r = 5m + 2$ then

$$5l^2 - (4r - 1)l + r^2 - r = 5(l - 2m - 1)^2 + 3(l - 2m - 1) + 5m^2 + m$$

and so

$$a_{5m+2}(q) = -(-1)^m q^{5m^2+m} \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+3n} = -(-1)^m q^{5m^2+m} \Omega(5n^2+3n).$$

If $r = 5m + 3$ then

$$5l^2 - (4r - 1)l + r^2 - r = 5(2m + 1 - l)^2 + (2m + 1 - l) + 5m^2 + 3m$$

and so

$$a_{5m+3}(q) = (-1)^m q^{5m^2+3m} \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+n} = (-1)^m q^{5m^2+3m} \Omega(5n^2+n).$$

If $r = 5m + 4$ then

$$5l^2 - (4r - 1)l + r^2 - r = 5(l - 2m - 2)^2 + 5(l - 2m - 2) + 5m^2 + 5m + 2$$

and so

$$a_{5m+4}(q) = (-1)^m q^{5m^2+5m+2} \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+5n}.$$

But in this sum the terms for $n = j$ and $n = -j - 1$ cancel and so $a_{5m+4}(q) = 0$. It follows that

$$\begin{aligned} P &= \Omega(5n^2 + n) \sum_{m=-\infty}^{\infty} (-1)^m (q^{5m^2-3m} x^{5m} + q^{5m^2+3m} x^{5m+3}) \\ &\quad - \Omega(5n^2 + 3n) \sum_{m=-\infty}^{\infty} (-1)^m (q^{5m^2-m} x^{5m+1} + q^{5m^2+m} x^{5m+2}) \\ &= \Omega(5n^2 + n) (\Omega(5n^2 - 3n, 5n) + \Omega(5n^2 + 3n, 5n + 3)) \\ &\quad - \Omega(5n^2 + 3n) (\Omega(5n^2 - n, 5n + 1) + \Omega(5n^2 + n, 5n + 2)) \end{aligned}$$

which establishes Theorem 1.

Remark After submitting this manuscript, the author received a copy of [3] which contains a proof of an assertion equivalent to Theorem 1, using a similar method.

References

- [1] H. M. Farkas & I. Kra, ‘On the quintuple product identity’, *Proc. Amer. Math. Soc.* **27** (1999), 771–778.

- [2] D. Foata & G.-H. Han, ‘The triple, quintuple and septuple product identities revisited’, *Séminaire Lotharingien de Combinatoire*, B42o (1999), 12 pp.
- [3] M. D. Hirschhorn, ‘An identity of Ramanujan, and applications’, *Contemp. Math.*, to appear.